Confinement, screening and the center on $S^3 \times S^1$

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ABSTRACT: We compute the one-loop effective potential for the Polyakov loop on $S^3 \times S^1$ for an asymptotically free gauge theory of arbitrary group G and a generic matter content. We apply this result to study the phase structures of G_2, SO(N) and Sp($2N$) gauge theories which turn out to be in qualitative agreement with the results of lattice calculations. On $S^3 \times S^1$, at zero temperature, the Polyakov loop is zero for kinematical reasons. For small but non-zero temperature, the Polyakov loop is still zero if the gauge theory has an unbroken center, while it acquires a small vacuum expectation value for gauge theories whose center is trivial or explicitly broken by the presence of dynamical matter fields. At high temperatures, the saddle point structure of the effective potential is different from the low temperature case suggesting that the theory is in the deconfined phase. At finite N, the Polyakov loop is non-zero in the high temperature phase only in theories with no unbroken center symmetry, consistent with screening of the external charge introduced by the Polyakov loop and Gauss law on a compact space.

KEYWORDS: 1/N Expansion, Lattice Gauge Field Theories, Confinement.

Contents

1. Introduction

Quantum Chromodynamics (QCD) is a confining theory: the elementary particles of the theory, quarks and gluons, never appear as final states of strong interactions. This property holds for temperatures below a critical value T_c, above which the system is deconfined, and quarks and gluons form a plasma. Confinement is a property of both QCD and the pure gauge theory without any matter. To gain a better understanding of this phenomenon, it is useful to study confinement in pure Yang-Mills theory. For pure SU(N) gauge theory, the existence of a deconfinement phase transition has been rigorously proved in [1].

The finite temperature theory is formulated on a four dimensional Euclidean space with one compact dimension. The length β of the compact dimension is the inverse of the temperature T. In the confined phase, the free energy of an isolated static quark is infinite. The free energy F is related to the vacuum expectation value of the Polyakov loop

$$L_R(\vec{x}) = \text{tr}_R \left(e^{\int_0^{1/T} A_0(\vec{x},t)dt} \right) , \tag{1.1}$$

where A_0 is the vector potential in the compact direction (parameterized by the coordinate t). Notice that A_0 is anti-Hermitian in this expression. A coupling constant factor g has been absorbed in the field as well. This means that in the classical Lagrangian the fields will not be canonically normalized, but there is a $1/g^2$ overall factor. The trace is taken in the representation R under which the quark transforms[1] and $\vec{x}$ is the position of the

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[1]In QCD, this is the fundamental representation; it is useful however to study a more general case, in which the quark is in a generic representation of the gauge group.

quark. If we define the vacuum expectation value (*vev*) of L (we drop the representation index from now on) as

$$\langle L \rangle = \frac{1}{V} \left\langle \int L(\vec{x}) d^3 x \right\rangle , \qquad (1.2)$$

the relationship between $L(\vec{x})$ and the free energy of the quark is given by [2]

$$F = -T \log \langle L \rangle . \qquad (1.3)$$

In SU(N) gauge theories, the behavior of $\langle L \rangle$ across the deconfinement phase transition can be explained in terms of a symmetry. This is the symmetry of the system under the $\mathbb{Z}_N$ center of the group, which leaves the Lagrangian unchanged. Under a transformation $z \in \mathbb{Z}_N$, $\langle L \rangle \to z \langle L \rangle$. In the deconfined phase, F is finite, i.e. $\langle L \rangle \neq 0$. This implies that in the deconfined phase the center symmetry is broken. Conversely, in the confined phase F is infinite, $\langle L \rangle = 0$ and the center symmetry is not broken. This leads to the natural interpretation that confinement reflects a change in the property of the vacuum under the $\mathbb{Z}_N$ center symmetry, the *vev* of the Polyakov loop being the order parameter for the transition. This observation lead Svetitsky and Yaffe to the well-known conjecture [3] that relates the deconfinement phase transition in a four dimensional SU(N) gauge theory to the order-disorder transition in three dimensional $\mathbb{Z}_N$ spin systems: since the properties of the deconfinement transition are determined by the underlying $\mathbb{Z}_N$ symmetry, the gauge and the corresponding spin system are in the same universality class when both transitions are second order.

Since confinement is a non-perturbative phenomenon, a fully non-perturbative approach such as putting the theory on a space-time lattice is mandatory for quantitative studies. For SU(N) gauge theory, recently it has been shown in [4] that important insights about the physics of color confinement and the phase structure of the theory can be obtained by studying the system on a $S^3 \times S^1$ manifold, where the radius β of the S^1 is connected to the temperature in the usual way ($\beta = 1/T$) and the radius R of the S^3 provides the IR cutoff at which the running of the gauge coupling freezes. When $1/R$ is much larger than the dynamical scale of the theory, the coupling is small and perturbative calculations are reliable. It is then possible to compute an effective potential for the Polyakov loop at one loop order in perturbation theory. Evaluating the partition function of the system for $\beta \ll R$ (high temperature regime) and $\beta \gg R$ (low temperature regime) yields information about the phase structure of the theory. In confining gauge theories, for a finite number of degrees of freedom a cross-over between the confined and deconfined phase is observed, with the two phases being characterized by a different structure of the minima of the free energy (or, using a more field theoretical terminology, the *effective potential*). In order to verify the existence of a real phase transition at some critical value of the temperature, an infinite number of degrees of freedom is needed. In standard thermodynamical calculations, this is achieved by taking the infinite volume limit of the system. This limit is obviously not possible for $S^3 \times S^1$, since the calculations rely on R being small. Instead, the thermodynamic limit is achieved here in another way: by sending to infinity the number of elementary degrees of freedom of the theory (one example in SU(N) gauge

theory is to take the limit of infinite number of colors N with fixed $g^2 N$ [5]). Thus, the existence of a phase transition can be rigorously proved only in some large N limit; for finite N or for exceptional groups, weak coupling calculations on a $S^3 \times S^1$ manifold can only discover the existence of two regimes (confined and deconfined), but cannot establish whether they are separated by a cross-over or a real phase transition in infinite volume.

Weak coupling calculations on $S^3 \times S^1$ have been used for determining the phase structure of various supersymmetric and non-supersymmetric gauge theories [6–10], treating each case independently. In this paper, we shall show that a general expression can be derived from the representation theory of the Lie Algebra of the gauge group that allows one to compute the one-loop effective potential for a generic gauge theory containing bosons and fermions in arbitrary representations. Using our result, it is straightforward to work out the phase structure of non-Abelian gauge theories for any gauge group and given matter content. We shall show that our computation reproduces the previously known results and we will use it to determine the phase structure of other gauge theories that can help us to understand the physics of color confinement, like G_2 gauge theory, with no center, and SO(N) gauge theory, whose center is either trivial (N odd) or $\mathbb{Z}_2$ (N even).[2] While a priori it is not obvious that the small volume calculations reproduce features of the thermodynamical limit, comparisons with other techniques (such as the lattice, as we will show in this paper) provide evidence that this is indeed the case.

Our calculation show that in general the minimum structure of the effective potential is different at low and high temperatures. This means that a cross-over (in the finite volume case) or possibly a phase transition (which can arise only if the thermodynamical limit can be taken, e.g. by sending to infinity the number of elementary degrees of freedom of the theory) separates those two regimes, which are naturally identified respectively with the confined and deconfined phase. However, in both regimes the details of the minimum structure depend on the center symmetry. A symmetry cannot be spontaneously broken on finite volume, so if there is a non-trivial center that is not broken explicitly by the matter content of the theory, the Polyakov loop cannot develop an expectation value at any temperature. The reason why the Polyakov loop vanishes in these theories differs in the low temperature and the high temperature phase. In the low temperature phase, the effective potential for the Polyakov loop will have only one saddle point in which the Polyakov loop will be zero. At high temperatures, there are multiple saddle points of the effective potential, each related by the action of the center. In each of the saddle points, the Polyakov loop is non-zero. However, since we are on finite volume, these saddle points are not superselection sectors and therefore we have to sum over all of them, which yields a zero value for the Polyakov loop. It is plausible to assume that the existence of the saddle points at high temperatures survives the infinite volume limit in which, we can restrict ourselves to one of the multiple saddle points, thereby breaking the center symmetry and signaling a confinement/deconfinment transition. The situation is different when the theory has no unbroken center subgroup. In that case, at both low and high temperatures, there is one global minimum of the effective potential. In the low temperature phase, the Polyakov

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[2]In this paper, we will exclusively study SO(N) groups and not their covering groups, Spin(N).

loop is small ($\mathcal{O}(e^{-\beta/R})$) and is zero at zero temperature whereas in the high temperature phase, we find that it develops an $\mathcal{O}(1)$ non-zero expectation value. This can be understood in terms of screening of the external charge introduced by the Polyakov loop. For theories with a non-trivial center, the external charge cannot be screened by gluons and quarks popping out of the vacuum. However, if there is no center symmetry, any external charge can be screened by an appropriate number of gluons and quarks.

The paper is organized as follows. In section 2 we compute the effective potential for a generic Lie group on a $S^3 \times S^1$ manifold and derive the behavior of the Polyakov loop at high and low temperature in terms of the root and weight vectors of the group and the matter content of the theory, emphasizing how our result reproduces previous calculations and how it can be used to analyze other gauge theories. In section 3, we specialize to the G_2 case, showing that the Polyakov loop is zero at sufficiently low temperature and different from zero at high temperatures. The same result is worked out for SO(N) gauge theories in section 4. In section 5 we show that our results are compatible with the non-Abelian Gauss law. Finally, section 6 sums up our findings and their implications for our understanding of color confinement.

2. Confinement on S^3 for an arbitrary gauge theory

In the canonical ensemble, the thermal partition function of a quantum field theory is equal to the Euclidean path integral of the theory with a periodic time direction of period $\beta = \frac{1}{T}$ and anti-periodic boundary conditions for the fermions along the temporal circle. We will consider a gauge theory with a semi-simple group G with matter in arbitrary representations on a spatial S^3. In the following, we will assume that the theories we want to study are asymptotically free. To understand the phase structure, we need to calculate the free energy which is just the log of the partition function. Therefore, we perform a Euclidean path integral for the theory on $S^3_R \times S^1_\beta$.

We take $1/R$ to be large compared to the dynamical scale of the theory , so the theory is at weak coupling. The S^1 topology allows non-trivial flat connections $F_{\mu\nu} = 0$ that can be parameterized in a gauge invariant way by the value of the Polyakov loop. The set of flat connections are the lightest degrees of freedom of the theory, and other excitations are separated in energy by a gap proportional to $1/R$. In fact, the flat connection is a zero mode of the theory, in the sense that the quadratic part of the action does not depend on it. The low energy phase structure can be studied using a Wilsonian approach where heavier degrees of freedom are integrated out to give an effective action for the flat connection. In the weak coupling regime, we can perform a perturbative expansion in the coupling and the one loop result provides a good approximation to the effective potential. Our discussion will generalize previous results [11, 12, 7].

We can use time-dependent gauge transformations to fix A_0 to be constant and in the Cartan subalgebra,

$$A_0 = \sum_{a=1}^{r} H^a C^a, \quad [H^a, H^b] = 0 \ \forall a, b \tag{2.1}$$

where r is the rank of the group. The classical value of the Polyakov loop is

$$L = \text{tr} \exp\left(\int_{S^1} A_0\right). \tag{2.2}$$

We will now consider adding matter fields to the gauge theory in arbitrary representations that can be obtained as tensor products of the fundamental (and anti-fundamental in case it exists). The fundamental representation corresponds to a mapping of the group to the (complex) general linear group acting on a vector space of dimension F. So any representation can be mapped to a tensor (we are ignoring the spinor representations of $SO(N)$ groups). The tensor can have covariant or contravariant indices, and the adjoint action can be defined as (summation understood)

$$\text{Ad}(A)\,X = A_{i_1}^{k_1} X_{k_1 i_2 \cdots i_n}^{j_1 \cdots j_m} + \cdots + A_{i_n}^{k_n} X_{i_1 \cdots k_n}^{j_1 \cdots j_m} - A_{l_1}^{j_1} X_{i_1 \cdots i_n}^{l_1 \cdots j_m} - \cdots - A_{l_m}^{j_1} X_{i_1 \cdots i_n}^{l_1 \cdots l_m}. \tag{2.3}$$

For $SU(N)$ groups, the Cartan subalgebra admits a diagonal form in terms of the fundamental weights $\vec{\nu}_i$, $i = 1, \ldots, F$, where F is the dimension of the fundamental representation.

$$H_{ij}^a = \nu_i^a \delta_{ij}. \tag{2.4}$$

The fundamental weights need not be linearly independent; this will happen only if $F = r$, the rank of the group. Real groups like $SO(N)$ and $Sp(N)$ cannot be diagonalized using gauge transformations. However, once we have fixed the flat connection to be constant, we can do a global similarity transformation to diagonalize it. Gauge invariant operators remain unchanged, so this transformation is allowed. In the case of $SO(N)$ this is done by a $U(N)$ transformation, while in the case of $Sp(N)$, it is a $U(2N)$ transformation. In the diagonal basis, the adjoint action simplifies to

$$\text{Ad}(A)\,X = \vec{C} \cdot \left(\sum_{a=1}^{n} \vec{\nu}_{i_a} X_{i_1 \cdots i_a \cdots i_n}^{j_1 \cdots j_m} - \sum_{b=1}^{m} \vec{\nu}_{j_b} X_{i_1 \cdots i_n}^{j_1 \cdots j_b \cdots j_m}\right), \tag{2.5}$$

which is

$$\text{Ad}(A)\,X = \vec{C} \cdot \left(\sum_{a=1}^{n} \vec{\nu}_{i_a} - \sum_{b=1}^{m} \vec{\nu}_{j_b}\right) X_{i_1 \cdots i_n}^{j_1 \cdots j_m} = (\vec{C} \cdot \vec{\mu}_{m,n}) X_{i_1 \cdots i_n}^{j_1 \cdots j_m}. \tag{2.6}$$

Here, $\vec{C}$, $\vec{\nu}$ and $\vec{\mu}_{m,n}$ are r dimensional vectors and the dot product is the standard Euclidean product. The weights of any representation are linear combinations of the fundamental weights, so $\vec{\mu}_{m,n} = (\sum_{a=1}^{n} \vec{\nu}_{i_a} - \sum_{b=1}^{m} \vec{\nu}_{j_b})$ is a weight of the tensor representation with m covariant and n contra-variant indices.

The covariant derivative acting on a field in the background field of the Polyakov loop is

$$D_\mu X = \partial_\mu X - \delta_{\mu 0} \text{Ad}(A)\,X. \tag{2.7}$$

To calculate the one-loop effective potential, we expand the gauge field, matter and the ghosts in harmonics on $S^3 \times S^1$ and keeping only the quadratic terms, we integrate out the various modes. The one loop potential will be in general

$$V(\vec{C}) = -\frac{1}{2} \sum_R (-1)^{F_R} \text{Tr}_{S^3 \times S^1} \log\left(-D_{0R}^2 - \Delta\right), \tag{2.8}$$

where R denotes the gauge group representation of the different fields we have integrated out, F_R is the fermionic number and the trace is over the physical degrees of freedom (zero modes excluded). This can be written as

$$V(\vec{C}) = \sum_R (-1)^{F_R} \sum_{\vec{\mu}_R} V_R(\vec{\mu}_R \cdot \vec{C}), \qquad (2.9)$$

where $\vec{\mu}_R$ are the weights of the representation and the potential V_R can depend on the Lorentz representation of the field. In flat space ($T^n \times R^d$ case) V_R is just an universal function times the number of physical polarizations (see [7], for instance). On S^3 the situation is different because the Kaluza-Klein decomposition depends on the spin of the field. The one-loop potential for several SU(N) theories on $S^3 \times S^1$ has been computed previously [4, 8]. We will generalize the computation to other groups and cast the results in a simple form.

In general,

$$\text{Tr}_{S^3 \times S^1} \log\left(-D_{0R}^2 - \Delta\right) = \sum_{\vec{\mu}_R} \sum_{\ell} \sum_{n=-\infty}^{\infty} \mathcal{N}_R(\ell) \log\left(\varepsilon_\ell^2 + \left(\frac{2\pi n}{\beta} + \vec{\mu}_R \cdot \vec{C}\right)^2\right), \quad (2.10)$$

where $\mathcal{N}_R$ is the number of physical polarizations of the field and ε_ℓ is the energy of the Kaluza-Klein mode ψ_l satisfying $\Delta\psi_l = -\varepsilon_\ell^2 \psi_l$ (Δ is the Laplacian on S^3). $\mathcal{N}_R$ and ε_ℓ depend on the field type that is being integrated out in a way that we list below.

(i) Scalars. Scalar field can be conformally or minimally coupled. The set of eigenvectors is the same in both cases. However, their eigenvalues are different. For conformally coupled scalars[3] we have $\varepsilon_\ell = R^{-1}(\ell+1)$ whereas for minimally coupled scalars $\varepsilon_\ell = R^{-1}\sqrt{\ell(\ell+2)}$. In both cases, there is a degeneracy $\mathcal{N}_R(\ell) = (\ell+1)^2$ with $\ell \geq 0$.

(ii) Spinors. For 2-component complex spinors, we have $\varepsilon_\ell = R^{-1}(\ell+1/2)$ and $\mathcal{N}_R(\ell) = 2\ell(\ell+1/2)$ with $\ell > 0$.

(iii) Vectors. A vector field V_i can be decomposed into the image and the kernel of the covariant derivative: $V_i = \nabla_i \chi + B_i$, with $\nabla^i B_i = 0$. The eigenvectors for B_i, have $\varepsilon_\ell = R^{-1}(\ell+1)$ and $\mathcal{N}_R(\ell) = 2\ell(\ell+2)$ with $\ell > 0$. On the other hand, the part $\nabla_i \chi$ has $\varepsilon_\ell = R^{-1}\sqrt{\ell(\ell+2)}$ with degeneracy $\mathcal{N}_R(\ell) = (\ell+1)^2$ but with $\ell > 0$ only.

Notice that both spinors and vectors have no zero ($\ell = 0$) modes on S^3. This is why in a pure gauge theory the only field with a zero mode is A_0 which is a scalar on S^3.

It is a standard calculation using Poisson resummation to show that up to an infinite additive constant,

$$\text{Tr}_{S^3 \times S^1} \log\left(-D_{0R}^2 - \Delta\right) = \sum_{\vec{\mu}_R} \sum_{\ell \geq \ell_{\min}}^{\infty} \mathcal{N}_R(\ell)\left\{\beta\varepsilon_\ell - \sum_{n=1}^{\infty} \frac{1}{n} e^{-n\beta\varepsilon_\ell} \cos(n\beta\,\vec{\mu}_R \cdot \vec{C})\right\}. \quad (2.11)$$

Where $\ell_{\min} = 0, 1$ depending on whether the field has constant modes on the S^3 or not. The first term here involves the Casimir energy and since it is independent of $\vec{C}$ will play

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[3]These have a mass term involving the Ricci scalar of the manifold, in this case simply R^{-1}.

no role in our analysis and we will subsequently drop it. Ignoring this term, the potential V_R is given by

$$V_R(x) = 2 \sum_\ell \mathcal{N}_R(\ell) \sum_{n=1}^\infty e^{-n\beta\varepsilon_\ell} \frac{\sin^2\left(\frac{n\beta}{2}x\right)}{n}. \tag{2.12}$$

The sum on n can be done explicitly and the result is, up to a constant

$$V_R(x) = \frac{1}{2} \sum_\ell \mathcal{N}_R(\ell) \log\left(1 + \frac{\sin^2\left(\frac{\beta x}{2}\right)}{\sinh^2\left(\frac{\beta\varepsilon_\ell}{2}\right)}\right) \tag{2.13}$$

This is the contribution to the effective potential obtained by integrating out a particular field. The information about the type of field being integrated out (its Lorentz transformation properties etc) is encoded in $\mathcal{N}_R(\ell)$ and ε_ℓ as explained above.

In the remainder of this section, we will restrict ourselves to a pure gauge theory with no matter fields. Then, the basic degree of freedom is the non-Abelian gauge potential A_μ which decomposes into a time component A_0 which transforms as a minimally coupled (massless) scalar on S^3 and a vector A_i on S^3. We can write $A_i = \nabla_i\chi + B_i$ such that $\nabla_i B^i = 0$. Gauge fixing $\left(\nabla_i A^i + D_0 A_0\right) = 0$ by the usual Fadeev-Popov procedure leads to the existence of ghosts.

Then, integrating out the A_0, B_i, χ and ghost modes can be performed using the following table:

field	S^3 KK $-$ mode	ε_ℓ	$\mathcal{N}_R(\ell)$
B_i	$\ell > 0$	$(\ell+1)/R$	$2\ell(\ell+2) \times (+1)$
χ	$\ell > 0$	$\sqrt{\ell(\ell+2)}/R$	$(\ell+1)^2 \times (+1)$
ghosts	$\ell \geq 0$	$\sqrt{\ell(\ell+2)}/R$	$(\ell+1)^2 \times (-2)$
A_0	$\ell \geq 0$	$\sqrt{\ell(\ell+2)}/R$	$(\ell+1)^2 \times (+1)$

Notice that for $\ell > 0$, the modes χ, ghost and A_0 cancel and the only contributions is from divergenceless modes, B_i. The constant $\ell = 0$ ghost modes and A_0 modes give the following contribution:

$$V(x) \sim -\sum_{n=1}^\infty \frac{\cos(nx)}{n} \sim \log\sin^2\left(\frac{x}{2}\right) \tag{2.14}$$

This contribution is precisely the logarithm of the Jacobian that converts the integrals over $\vec{C}$ into the invariant integral over the group element $g = \exp\left(i\vec{\mu}_{\text{Ad}}.\vec{C}\right)$, yielding the correct gauge invariant measure in the path integral $d\mu(g)$, $g \in G$.

$$\mathcal{Z} \simeq \int d\vec{C} \prod_{\mu_{\text{Ad}}} \sin\left(\frac{\vec{\mu}_{\text{Ad}} \cdot \vec{C}}{2}\right) \cdots \simeq \int d\mu(g) \cdots . \tag{2.15}$$

The contribution from the divergenceless part of A_i which we have been denoting by B_i can be easily written down using (2.13) and reading off the corresponding ε_ℓ and $\mathcal{N}_R(\ell)$ from the table above. Then, the total effective potential is

$$V_{\text{gauge}}(\vec{C}) = \sum_{\vec{\mu}_{\text{Ad}}} \mathcal{V}(\vec{\mu}_{\text{Ad}} \cdot \vec{C}), \tag{2.16}$$

where

$$\mathcal{V}(x) = -\log\left|\sin\left(\frac{x}{2}\right)\right| + \sum_{\ell > 0} \ell(\ell+2)\log\left(1 + \frac{\sin^2\left(\frac{x}{2}\right)}{\sinh^2\left(\frac{\beta(\ell+1)}{2R}\right)}\right). \qquad (2.17)$$

The two terms in the effective potential come with opposite signs: the first term corresponds to a log repulsion coming from the $\ell = 0$ modes whereas the second term is an attractive term. It is the interplay between these two types of contributions that leads to an interesting phase structure for the theory.

In the low-temperature limit $\beta/R \gg 1$, the attractive contribution to the potential is exponentially supressed

$$\mathcal{V}(x) \simeq -\log\left|\sin\left(\frac{x}{2}\right)\right| + \sum_{\ell > 0} 4\ell(\ell+2)\sin^2\left(\frac{x}{2}\right)e^{-\frac{\beta(\ell+1)}{R}} + \dots. \qquad (2.18)$$

In the high-temperature limit $\beta/R \ll 1$, the lower mode terms become stronger and cancel the repulsive term except very close to the origin $x \sim \beta/R$

$$\mathcal{V}(x) \simeq -\log\left|\sin\left(\frac{x}{2}\right)\right| + 6\log\left|\sin\left(\frac{x}{2}\right)\right| + \dots. \qquad (2.19)$$

Therefore, we expect a change of behavior with increasing temperature. At low temperatures the expectation value of the Polyakov loop is zero up to exponentially supressed contributions. This can be seen from the fact that the repulsive term becomes the group measure (2.15) and, since the Polyakov loop corresponds to the character of the group associated to the representation R,

$$\langle L \rangle = \frac{1}{\mathcal{Z}} \int d\mu(g)\chi_R(g) = 0. \qquad (2.20)$$

In the zero temperature limit this is a kinematical statement: there cannot be unscreened charges in a compact space. Therefore, confinement is just a kinematical fact. The derivation of the Haar measure in $S^3 \times S^1$ was known for unitary groups [4], but our discussion is completely general and includes any semi-simple Lie group.

It is interesting to notice that at higher temperatures the expectation value of the Polyakov loop could be non-zero *even in finite volume*. An intuitive explanation follows from considering the Polyakov loop as the insertion of an external source in the system. At zero temperature pair production from the vacuum is completely frozen out, so it will cost an 'infinite amount of energy' to introduce a charge in finite volume. However, at higher temperatures the system is in an excited state where the charges present in the medium can screen the external source. Whether this actually happens or not depends on the details of the group and the type of source we introduce. In gauge theories with unitary groups this issue boils down to whether the center of the group is broken or not. In pure Yang-Mills the center is unbroken, so the expectation value of the Polyakov loop should be zero at any temperature. On the other hand, in a theory with quarks the center is completely broken, so the Polyakov loop can be nonzero at high temperatures.

In general, the lack of screening implies that theories with an unbroken center have a vanishing Polyakov loop at all temperatures. However, at high temperatures the main

contributions to the path integral will come from a sum over saddle points corresponding to the minima of the potential. For very large temperatures the measure can be approximated by a sum of delta functions, so the value of the Polyakov loop is just the sum at the different minima. At each of these points the Polyakov loop has a nonzero value and it is plausible to assume that there is a map between them and the possible vacua of the gauge theory in the deconfined phase in infinite volume, where kinematical constraints do not apply. In this sense, by restricting ourselves to only one saddle point at high temperatures thereby getting a non-zero value of the Polyakov loop, we can interpret these results as a confinement-deconfinement phase transition at weak coupling.

Strictly speaking there is no transition at finite volume, in the high temperature phase we should sum over all the vacua so the average of the Polyakov loop will be still zero or we would go smoothly from a zero to a non-zero value as we increase the temperature. However, it is possible to show there is a sharp transition in the large N limit [4]. Of course this is not applicable to the exceptional groups but only to the classical groups.

2.1 Large N phase transition

In the ensuing analysis, it is convenient to write the potential in terms of the Polyakov loop eigenvalues $\lambda_i = \vec{\nu}_i \cdot \vec{C}$, where $\vec{\nu}_i$ are the fundamental weights. This immediately follows from the fact that the weights of any tensor representation are linear combinations of the fundamental weights, and the contributions to the effective potential coming from integrating out such representations will depend on corresponding linear combinations of λ_i. In the $N \to \infty$ limit for the classical groups, we can approximate the discrete set of eigenvalues by a continuous variable θ and substitute the sums over eigenvalues by integrals in θ

$$\sum_{\lambda_i} \to N \int d\theta \rho(\theta) \,, \tag{2.21}$$

with an eigenvalue density $\rho(\theta) \geq 0$, such that

$$\int_{-\pi}^{\pi} d\theta \rho(\theta) = 1 \,. \tag{2.22}$$

The domain of integration is determined by the periodicity of the Polyakov loop eigenvalues. The potential becomes a functional of the eigenvalue distribution, and the system is well described by the saddle point that minimizes the potential. When the parameters of the theory change, such as the ratio β/R, the dominating saddle point can change, leading to a phase transition.

For pure $\mathrm{U}(N)$ Yang-Mills, the potential is given by equation (2.17) with $x = \vec{\mu}_{\mathrm{adj}}.\vec{C} = \lambda_i - \lambda_j$. After converting the summation to the integral via the density of eigenvalues, we obtain

$$V[\rho] = N^2 \int_{-\pi}^{\pi} d\theta \rho(\theta) \int_{-\pi}^{\pi} d\theta' \rho(\theta') \sum_{k=1}^{\infty} \frac{1}{k} (1 - z_V(q^k)) \cos(k(\theta - \theta')) \tag{2.23}$$

where $q = e^{\beta/R}$ and z_V is the partition function of divergenceless vector fields in S^3

$$z_V(q) = \sum_{\ell > 0} 2\ell(\ell + 2) q^{-(\ell+1)} = 2 \frac{3 \, q - 1}{(q - 1)^3} \,. \tag{2.24}$$

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We can expand the eigenvalue distribution in complex Fourier components ρ_n

$$\rho(\theta) = \frac{1}{2\pi} + \sum_{n=1}^{\infty} \text{Re}(\rho_n) \cos(n\theta) + \sum_{n=1}^{\infty} \text{Im}(\rho_n) \sin(n\theta) \qquad (2.25)$$

and perform the integrals. We are left with a quadratic potential for each Fourier component

$$V = N^2 \pi^2 \sum_{n=1}^{\infty} \frac{(1 - z_V(q^n))}{n} |\rho_n|^2 . \qquad (2.26)$$

For low enough temperatures, $z_V(q^n) < 1$, and for all n, the coefficients in the summation are positive so the configuration of minimal energy is the uniform distribution $\rho(\theta) = 1/(2\pi)$. In this case the expectation value of the Polyakov loop vanishes, giving a weak coupling version of the confining phase. When we increase the temperature, the coefficient of the $n = 1$ component first becomes zero at a critical value $q_c = 2 + \sqrt{3}$ and then negative beyond this point. If we continue increasing the temperature, eventually larger Fourier components become unstable beyond the points $q_c^{(n)} = q_c^{1/n}$ of marginal stability. At the critical value q_c the eigenvalue distribution jumps to a qualitatively different saddle point characterized by the appearance of a gap in the eigenvalue distribution, i.e. there is a range of values on θ over which that function $\rho(\theta)$ has no support. At temperatures just above q_c,

$$\rho(\theta) = \frac{1}{2\pi}(1 + \cos(\theta)) . \qquad (2.27)$$

The expectation value of the Polyakov in this saddle point is no longer zero, so this transition is a weak coupling analog of the confinement/deconfinement phase transition.

2.2 Some examples

We would like to emphasize that the formulas derived in this section are completely general on $S^3 \times S^1$ for any semi-simple gauge group and for fields in any representation. We can cast the complete one-loop potential as a sum of different terms determined by the spin of the field and the weight system of the group representation. Possible examples are pure gauge theories, with unitary, orthogonal symplectic or exceptional groups. We will comment some of them in the next sections.

Here we give a small list of examples in terms of the eigenvalues of the Polyakov loop $\lambda_i = \vec{\nu}_i \cdot \vec{C}$ for unitary groups. The phase diagram of most of them has been analyzed previously in the large-N limit [6, 9], showing an interesting structure in terms of the expectation values of the Polyakov loop. All of them exhibit a deconfinement transition at some critical temperature T_c. We will use the potential generated by Weyl fermions

$$\mathcal{V}_F(x) = \frac{1}{2} \sum_{\ell > 0} \ell(2\ell + 1) \log \left(1 + \frac{\sin^2\left(\frac{x}{2} + \left(\frac{\pi}{2}\right)\right)}{\sinh^2\left(\frac{\beta(2\ell+1)}{4R}\right)} \right) . \qquad (2.28)$$

The $\pi/2$ addition in the argument is for anti-periodic boundary conditions on fermions. Then, we can write the following potentials:

- QCD: SU(N) Yang-Mills with N_f massless flavors (or theories with fundamental matter in general)

$$V(\{\lambda_i\}) = \sum_{i \neq j} \mathcal{V}(\lambda_i - \lambda_j) - 2N_f \sum_{i=1}^{N} \mathcal{V}_F(\lambda_i) .$$

At high temperatures, the flavor contribution to the potential favors the $\lambda_i = 0$ minimum over other center-breaking minima. At low temperatures the flavor potential is exponentially supressed, and only the repulsive term from the pure gauge contribution is relevant.

- Supersymmetric theories: $\mathcal{N} = 1$ SU(N) (or theories with adjoint matter in general)

$$V(\{\lambda_i\}) = \sum_{i \neq j} \mathcal{V}(\lambda_i - \lambda_j) - \sum_{i \neq j} \mathcal{V}_F(\lambda_i - \lambda_j) .$$

Notice that when $\ell \to \infty$, fermionic and bosonic contributions cancel at leading order if we choose periodic boundary conditions for fermions. At high temperatures all the center-breaking minima become deeper, so center symmetry is unbroken. At large N, the system will sit at one of these center-breaking minima. At low temperatures the fermionic contribution is exponentially supressed.

- Orientifold theories [13]: SU(N) gauge theories with a Dirac fermion in the anti-symmetric (or symmetric) representation (or theories with two-index representations in general)

$$V(\{\lambda_i\}) = \sum_{i \neq j} \mathcal{V}(\lambda_i - \lambda_j) - \sum_{i \neq j} \mathcal{V}_F(\lambda_i + \lambda_j) - \left(2 \sum_{i=1}^{N} \mathcal{V}_F(\lambda_i) \right) .$$

The term inside the parentheses only appears in the symmetric representation; in the large-N limit its contribution is sub-leading. At high temperatures there are two possible situations depending on N being even or odd. In the first case, there are two-minima that become deeper w.r.t other center minima. If N is odd, there is only one minimum. In general, the vacuum corresponds to non-trivial center vacua [12, 7]. At low temperatures we recover the pure gauge potential.

- Orbifold theories (c.f. [14]), for instance the $\mathbb{Z}_2$ case with SU(N) × SU(N) gauge group and fermions in the bifundamental representation (or theories with several gauge group factors in general)

$$V(\{\lambda_i^{(1)}\}, \{\lambda_i^{(2)}\}) = \sum_{i \neq j} \mathcal{V}\left(\lambda_i^{(1)} - \lambda_j^{(1)}\right) + \sum_{i \neq j} \mathcal{V}\left(\lambda_i^{(2)} - \lambda_j^{(2)}\right) - 2 \sum_{i,j} \mathcal{V}_F\left(\lambda_i^{(1)} - \lambda_j^{(2)}\right) .$$

In the limit $\ell \to \infty$ and large N, the diagonal sector $\lambda_i^{(1)} = \lambda_i^{(2)}$ becomes supersymmetric. At high temperatures the minima are on $\lambda_i^{(1)} = -\lambda_i^{(2)}$ [12], and in this sector, the potential is equivalent to the orientifold case.

- Theories with more exotic matter content, etc. For a SU(N) gauge theory, whenever we have a representation of non-trivial N-ality η_R, the center vacua will be shifted at high temperatures to a band with levels of η_R degeneracy and a splitting proportional to η_R/N [7]. At low temperatures any contribution is exponentially supressed unless conformally coupled scalars are included. In that case, there will be extra zero modes that can modify the low-temperature regime.

3. The G_2 case

The results of section 2 can be applied to a gauge theory with exceptional group G_2. We briefly summarize its relevant group theoretic properties (we refer to [15] for a more in-depth discussion of G_2 gauge theory). G_2 has rank 2, so the potential (2.16) is defined over a two-dimensional plane. The fundamental representation is 7 and the adjoint 14 dimensional. The group SU(3) is a subgroup of G_2. The adjoint of G_2 under this embedding splits into an adjoint and the fundamental and anti-fundamental representations of SU(3): $\mathbf{14} \to \mathbf{8} \oplus \mathbf{3} \oplus \mathbf{3^*}$. Therefore, the roots of G_2 are the same as the roots and the weights of the fundamental and anti-fundamental representation of SU(3).

The fundamental representation of G_2 can be decomposed in terms of SU(3) representations as $\mathbf{7} \to \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{3^*}$. Then, the Polyakov loop of G_2 can be written in terms of a SU(3) matrix U

$$L = \text{tr} \begin{pmatrix} 1 & & \\ & U & \\ & & U^+ \end{pmatrix} = 1 + 2\,\text{Re}\,\text{tr}\,U\,. \tag{3.1}$$

As explained in section 2, we choose to work in the gauge where the SU(3) matrix for flat connections is parameterized by the constant Cartan subalgebra of the group

$$U = \text{diag}\left(e^{i\vec{\nu}_1 \cdot \vec{C}},\ e^{i\vec{\nu}_2 \cdot \vec{C}},\ e^{i\vec{\nu}_3 \cdot \vec{C}}\right)\,, \tag{3.2}$$

where the weights of the fundamental representation of SU(3), $\vec{\nu}^i$, are not linearly independent but satisfy the following conditions:

$$\vec{\nu}_i \cdot \vec{\nu}_j = \delta_{ij} - \frac{1}{3}\,, \quad \sum_{i=1}^{3} \vec{\nu}_i = 0\,. \tag{3.3}$$

The physical region of the $\vec{C}$ plane is obtained quotienting by global periodic transformations, that shift the phase of the eigenvalues of the Polyakov loop by multiples of 2π: $\vec{\nu}_i \cdot \vec{C} \to \vec{\nu}_i \cdot \vec{C} + 2\pi n_i$, and by Weyl transformations, that permute the eigenvalues of the Polyakov loop. In both cases, the root lattice determines the structure of the Cartan torus. Periodic transformations can be seen as translations along the roots $\vec{\alpha}_{ij} = \vec{\nu}_i - \vec{\nu}_j$, while Weyl transformations are seen as the reflections that move fundamental weights among themselves. The fixed lines of Weyl reflections lie along the roots. As a basis for the Cartan torus we can choose the simple roots $\vec{\alpha}_\pm = (1/\sqrt{2}, \pm\sqrt{3/2})$. Since $\vec{\nu}_i \cdot \vec{\alpha}_\pm = 0, 1$, the period is 2π. If we consider other representations, the group of Weyl reflections will be
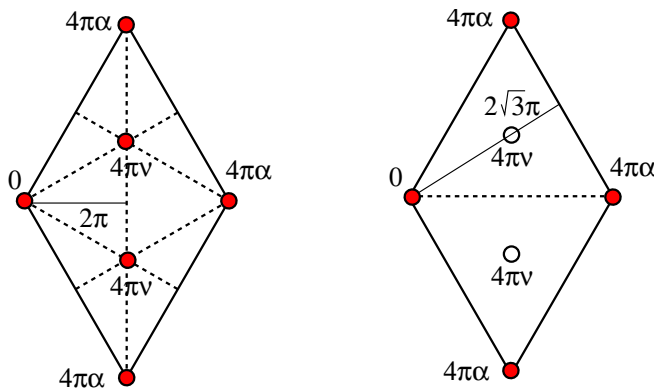
Figure 1: The Cartan torus (with roots normalized to unit norm) for SU(3) for the adjoint representation (on the left) and the fundamental representation (on the right). The Weyl lines are indicated by dashed lines. The global minima of the potential lie on the intersection of Weyl lines. For the fundamental representation, we also mark by white circles the smooth critical points of the potential. The center symmetry corresponds to translations on the torus that move the global minima of the adjoint representation among each other, notice that it is not a symmetry of the fundamental.
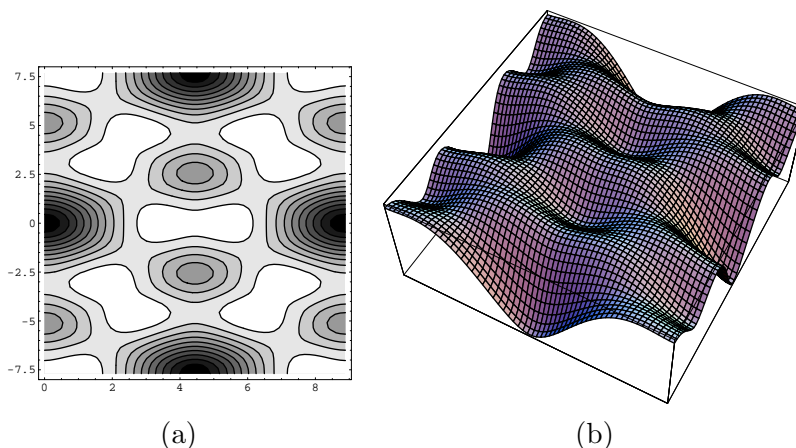

Figure 2: The potential for G_2 Yang-Mills at high temperatures. There is a global minima located at the origin of the Cartan torus and two local minima inside.

different and the associated Weyl lines will show a different structure. For a SU(3) theory with fields in the adjoint representation we can also make $\mathbb{Z}_3$ center transformations that shift the eigenvalues by multiples of $2\pi/3$. On the Cartan torus, this is seen as translations along the weights by 2π units. In figures 1, 2 we show some examples.

The potential is then

$$V(\vec{C}) = 2\sum_{i=1}^{3} \mathcal{V}\left(\vec{\nu}_i \cdot \vec{C}\right) + \sum_{i \neq j} \mathcal{V}\left((\vec{\nu}_i - \vec{\nu}_j) \cdot \vec{C}\right) . \tag{3.4}$$

In the SU(3) gauge theory, the first term in (3.4) does not appear. In that case, the attractive contribution of the potential has its minima located where the cosine is maximum, i.e.

at

$$(\vec{\nu}_i - \vec{\nu}_j) \cdot \vec{C} = 2\pi n \,, \quad n \in \mathbb{Z} \,. \tag{3.5}$$

For the adjoint representation, there are three minima located inside the physical region. In the pure Yang-Mills theory, one can move from one minimum to the other using $\mathbb{Z}_3$ center transformations. Therefore, the localization around the minima at high temperatures could be interpreted as the weak coupling analog of the breaking of the center in the infinite volume theory. When the temperature is very low, the repulsive contribution of the potential dominates, so the minima become maxima and we expect the configurations to be spread along the Cartan torus, giving a vanishing expectation value to the Polyakov loop. When we consider matter in the fundamental representation of SU(3), or in the G_2 pure gauge theory, the center is no longer a symmetry. In the one-loop potential (3.4), this is reflected by the presence of the first term. The minima of the attractive contribution are located at

$$\vec{\nu}_i \cdot \vec{C} = 2\pi n \,, \quad n \in \mathbb{Z} \,. \tag{3.6}$$

So there is only one minimum inside the physical region. Then, only one of the minima of the SU(3) adjoint contribution is a global minimum, while the other two can only be local minima at most. However, the localization around the minimum still occurs when we change the temperature.

In figure 3 we have plotted the G_2 potential for different temperatures. We notice that in the high temperature phase there is a single global minimum located at the corners of the physical region. This minimum corresponds to the trivial $A_0 = 0$ configuration, with $\operatorname{tr} U = 1$. There are also two local minima that in the pure SU(3) gauge theory will be the minima corresponding to the Polyakov loop values $\operatorname{tr} U = e^{\pm 2\pi i/3}$. When we lower the temperature, the minima become repulsive points due to the logarithmic term in the potential. The most repulsive point is the trivial configuration, so the configurations will be distributed around the center values.

We can compute the expectation value of the Polyakov loop at different temperatures. In terms of the eigenvalues $\lambda_i = \vec{\nu}_i \cdot \vec{C}$, the zero temperature $T = 0$ path integral is

$$\mathcal{Z}_0 = \left(\prod_{i=1}^{3} \int_{-\pi}^{\pi} d\lambda_i \right) \delta \left(\sum_{i=1}^{3} \lambda_i \right) \prod_{i<j} \sin^2 \left(\frac{\lambda_i - \lambda_j}{2} \right) \prod_{i=1}^{3} \sin^2 \left(\frac{\lambda_i}{2} \right) \equiv \int d\mu(\lambda) \,. \tag{3.7}$$

The expectation value of the Polyakov loop is exactly zero

$$\langle L \rangle_{T=0} = \frac{1}{\mathcal{Z}_0} \int d\mu(\lambda) \left(1 + 2 \sum_{i=1}^{3} \cos(\lambda_i) \right) = 0 \,. \tag{3.8}$$

At non-zero temperature $T = 1/\beta$, the Kaluza-Klein modes on S^3 give non-trivial contributions

$$\mathcal{Z}_\beta = \int d\mu(\lambda) \prod_{\ell=1}^{\infty} \left[\prod_{i<j} \left(1 + \frac{\sin^2 \left(\frac{\lambda_i - \lambda_j}{2} \right)}{\sinh^2 \left(\frac{\beta(\ell+1)}{2R} \right)} \right) \prod_{i=1}^{3} \left(1 + \frac{\sin^2 \left(\frac{\lambda_i}{2} \right)}{\sinh^2 \left(\frac{\beta(\ell+1)}{2R} \right)} \right) \right]^{-2\ell(\ell+2)} \,. \tag{3.9}$$

The expectation value becomes nonzero, and for high temperatures the measure is peaked around $\lambda_i = 0$. This qualitative picture agrees with recent lattice results [16].
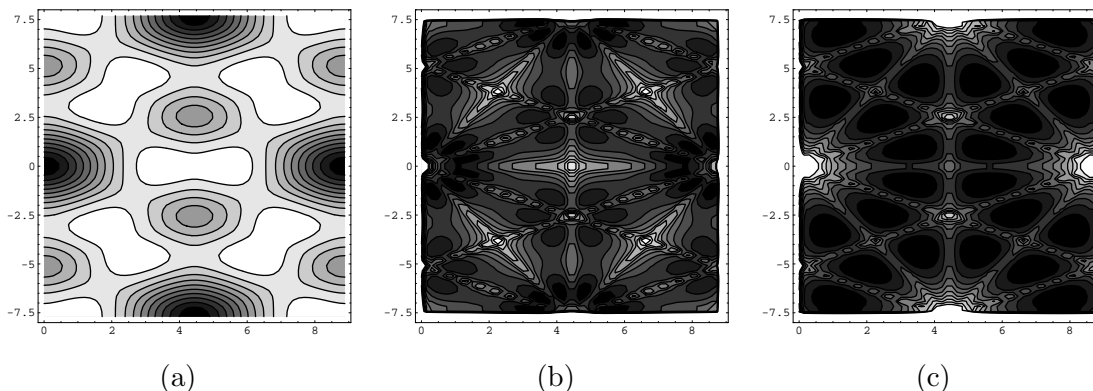
Figure 3: We show the G_2 one-loop potential on the Cartan torus for (a) high temperatures $\beta/R = 0.01$, (b) intermediate temperatures $\beta/R = 1.25$ and (c) low temperatures $\beta/R = 100$. As we lower the temperature, the initial single minimum $A_0 = 0$ becomes a maximum and new non-trivial symmetrically located minima appear, being pushed apart. The appearance of strong repulsion lines along the symmetry axis is due to the logarithmic term. These two features do not appear in other topologies, like the torus [7, 17]. Notice that the *vev* of the Polyakov loop would be an average over the Cartan torus weighted by the potential.

4. Orthogonal and symplectic groups

In this section, we will consider the phase structure of $SO(N)$ and $Sp(2N)$ gauge theories. The center of $Sp(2N)$ is $\mathbb{Z}_2$ and that of $SO(N)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$. for N even and $\mathbb{Z}_2$ for N odd. However, since we are not considering spinors, only a $\mathbb{Z}_2$ subgroup of the center will be relevant for our analysis for orthogonal groups for even N. For odd N, $SO(N)$ groups have a trivial center.

The non-trivial roots of $SO(2N)$ and $Sp(2N)$, in terms of the fundamental weights $\vec{\nu}$, are given by $\pm\vec{\nu}_i \pm \vec{\nu}_j$ (for $SO(2N)$, $i \neq j$ whereas there is no such restriction for $Sp(2N)$). For $SO(2N)$, using the gauge transformation, the Polyakov loop can be brought in the form:

$$
L = \begin{pmatrix}
\cos\theta_1 & \sin\theta_1 \\
-\sin\theta_1 & \cos\theta_1 \\
& & \cos\theta_2 & \sin\theta_2 \\
& & -\sin\theta_2 & \cos\theta_2 \\
& & & & \ddots & \ddots \\
& & & & & & \cos\theta_N & \sin\theta_N \\
& & & & & & -\sin\theta_N & \cos\theta_N
\end{pmatrix}, \tag{4.1}
$$

Also, by a gauge transformation we can flip the signs of θ's: $\theta_i \rightarrow -\theta_i$. The effective potential for the θ_i's is given by

$$
V(\{\theta_i\}) = \sum_{i \neq j} \mathcal{V}(\theta_i - \theta_j) + \mathcal{V}(\theta_i + \theta_j). \tag{4.2}
$$

Introducing the density of eigenvalues $\rho(\theta) = \frac{1}{2\pi} + \sum_{n=1}^{\infty} \rho_n \cos n\theta$, (where the Fourier coefficient ρ_n is now real and $\rho(\theta) = \rho(-\theta)$), and converting summations to integrals, we

obtain

$$V[\rho] = N^2 \int_{-\pi}^{\pi} d\theta \rho(\theta) \int_{-\pi}^{\pi} d\theta' \rho(\theta') \sum_{n=1}^{\infty} \frac{1}{n}(1 - z_V(q^n))(\cos(n(\theta - \theta')) + \cos(n(\theta + \theta'))). \quad (4.3)$$

Defining the Fourier components of $\rho(\theta)$ as in (2.25), the potential can be written as

$$V[\rho_n] = N^2 \pi^2 \sum_{n=1}^{\infty} \frac{(1 - z_V(q^n))}{n} \rho_n^2. \quad (4.4)$$

For low temperatures, $\rho_n = 0$, which implies a uniform distribution $\rho(\theta) = \frac{1}{2\pi}$ and a vanishing expectation value of the Polyakov loop. At extremely high temperatures, the arguments of the cosine on the right hand side of (4.3) have to be minimized, which happens when the θ_i' are the same and equal to 0 or π, i.e.

$$\rho_+(\theta) = \delta(\theta) \quad \text{or} \quad \rho_-(\theta) = \delta(\theta - \pi).$$

In the two saddle points, the Polyakov loop $\langle \operatorname{tr} O \rangle_{\pm} = 2N \int d\theta \, \rho_{\pm}(\theta) \cos(\theta)$ is

$$\langle \operatorname{tr} L \rangle_+ = +1 \qquad \rho_+(\theta) = \delta(\theta),$$
$$\langle \operatorname{tr} L \rangle_- = -1 \qquad \rho_-(\theta) = \delta(\theta - \pi).$$

If we sum over the different saddle points, we get a trivial value of the Polyakov loop. However, in the large N limit, we can choose one saddle point since large N provides a thermodynamic limit. In that case, the Polyakov loop is non-zero and the center symmetry is spontaneously broken.

The situation is different for $SO(2N + 1)$. The group $SO(2N + 1)$ has no center. The roots, in terms of the fundamental weights $\vec{\nu}$, are given by $\pm\vec{\nu}_i \pm \vec{\nu}_j$ and $\pm\vec{\nu}_i$. The Polyakov loop is now a $(2N + 1) \times (2N + 1)$ matrix similar to (4.1) with an extra row and column having 1 along the diagonal and zeros everywhere else. The effective potential for the θ_i's is now given by

$$V(\{\theta_i\}) = \sum_{i \neq j} \mathcal{V}(\theta_i - \theta_j) + \mathcal{V}(\theta_i + \theta_j) + 2 \sum_i \mathcal{V}(\theta_i). \quad (4.5)$$

As before, we can introduce a density of eigenvalues and its Fourier components yielding an effective potential (4.4). The effective potential then is

$$V[\rho] = N^2 \sum_{n=1}^{\infty} \frac{1}{n}(1 - z_V(q^n))\Big(\int_{-\pi}^{\pi} d\theta \rho(\theta) \int_{-\pi}^{\pi} d\theta' \rho(\theta')\big(\cos(n(\theta - \theta')) + \cos(n(\theta + \theta'))\big)$$
$$+ \frac{1}{N} \int_{-\pi}^{\pi} d\theta \rho(\theta) \cos(n\theta)\Big). \quad (4.6)$$

Defining the Fourier components of $\rho(\theta)$ as in (2.25), the potential can be written as

$$V[\rho_n] = N^2 \pi^2 \sum_{n=1}^{\infty} \frac{(1 - z_V(q^n))}{n}\left(\rho_n^2 + \frac{1}{\pi N}\rho_n\right). \quad (4.7)$$

This is minimized when $\rho_n = -\frac{1}{2\pi N}$. This corresponds to an eigenvalue distribution $\rho(\theta) = \frac{1}{2\pi}(1 + \frac{1}{2N}) - \frac{1}{2N}\delta(\theta)$. The Polyakov loop is given by

$$\langle \text{tr} L \rangle = 1 + 2N \int_{-\pi}^{\pi} d\theta \rho(\theta) \cos\theta = 1 + 2N \int_{-\pi}^{\pi} d\theta \left(\frac{1}{2\pi} \left(1 + \frac{1}{2N} \right) - \frac{1}{2N}\delta(\theta) \right) \cos\theta = 0. \quad (4.8)$$

In the extreme high temperature phase, because of the second line in (4.6) which picks out the $\theta = 0$ saddle point, $\rho(\theta) = \delta(\theta)$, i.e. there is a unique saddle point in this case, giving a non-zero value of the Polyakov loop. The linear term on ρ_n is similar to the one present for SU(N) theories with fundamental matter. In that case, the potential becomes

$$V[\rho_n] = N^2 \pi^2 \sum_{n=1}^{\infty} \frac{(1 - z_V(q^n))}{n} \rho_n^2 + \frac{N_f}{N} \frac{z_F(q^n)}{n} \rho_n, \quad (4.9)$$

and is minimized at

$$\rho_n = -\frac{N_f}{2N} \frac{z_F(q^n)}{1 - z_V(q^n)}. \quad (4.10)$$

The value of the Fourier components is bounded to $|\rho_n| \leq 1$ because $\rho(\theta) \geq 0$. The bound is saturated at a temperature below the Hagedorn temperature, and the eigenvalue density changes to a gapped distribution in a continuous way. It can be shown that the deconfinement transition becomes Gross-Witten like [18, 19]. However, in the case of SO($2N + 1$) theories, both the quadratic and the linear coefficient come from the adjoint contribution and are the same. Hence $\rho_n = -1/(2\pi N)$ is fixed and therefore the transition is Hagedorn-like. A picture of the phase transition in SO(3) in terms of non-trivial twist sectors of the theory has been presented in [20].

In the large N limit, the behavior of the theory with gauge group Sp($2N$) is identical the the SO($2N$) case.

5. Gauss law and screening

In the previous two sections, we find that the Polyakov loop can be non-zero at high temperature is theories with no surviving center symmetry, even on finite volume. We are introducing a non-Abelian charge in finite volume and it does not cost infinite free energy. This is consistent with the non-Abelian Gauss law which we can write as

$$\int_{S^3} \left(\nabla_i E_i^a - f^{abc} A_i^b E_i^c \right) = Q^a, \quad (5.1)$$

where E_i^a are the electric components of the non-Abelian field, A_i^a is the magnetic potential, f^{abc} are the group structure constants and Q^a is the total external charge. The first term on the left hand side is a total derivative and vanish when integrated over a compact space. This is the only term that is present for Abelian fields and implies the known result that we cannot introduce an Abelian charge on finite volume. However, for non-Abelian fields the second term can be non-zero and Gauss law can be satisfied in the presence of a net external charge.

For a gauge theory with group G and an unbroken center $\mathcal{C}$, both the measure and the action are invariant under center transformations. This implies that there is a *global*

center symmetry which cannot be spontaneously broken in finite volume. Then, the expectation value of any center-breaking operator should be zero. The Polyakov loop in any representation charged under the center transforms non-trivially under the center (which can be seen by performing large gauge transformations that are periodic up to an element of the center). Therefore, its expectation value should be zero, which corresponds to the statement that the free energy cost of introducing external particles that are charged under the center is infinite. When there is no center symmetry because the gauge group is centerless or the representation of the matter fields explicitly breaks it, there is no symmetry protecting the expectation value of the Polyakov loop and indeed we observe that it is nonzero in general.

In the zero temperature limit, we find that the Polyakov loop is zero for any theory. We can also understand this in terms of Gauss law. In this limit, we can ignore the Polyakov loop and the term coming from the non-Abelian nature of the group in Gauss law. This is because we are essentially working in the zero coupling limit. The only non-trivial interaction terms we have kept correspond to the interaction of the fields with the background field which is a massless mode of the theory $A_i = L(\vec{x})\partial_i L(\vec{x})$.[4] There is no background field at zero temperature and therefore no non-Abelian term in Gauss law. In this case, it costs infinite action to introduce any external charge and the Polyakov loop expectation value is zero.

This is consistent with the physical picture in terms of screening. In the absence of a surviving center symmetry, an external particle in any representation can be screened by an appropriate number of gluon or matter particles, which allows us to introduce them at finite free energy cost. In theories with a center symmetry, an external particle charged under the center cannot be screened and therefore cannot have finite free energy in finite volume.

6. Conclusions

In this paper, we have investigated confinement on $S^3 \times S^1$ for a gauge theory based on an arbitrary semi-simple Lie group containing matter in generic representations. Our analytic calculation mimics the behavior of the Polyakov loop computed by lattice simulations for any theory for which lattice data exist. [16, 17, 21 – 23].

By using the general expression for the one-loop effective potential, we have shown that at zero temperature the Polyakov loop is exactly zero for any theory. Since the Polyakov loop is naturally related to the free energy of an isolated quark, our result implies that on a $S^3 \times S^1$ manifold, any gauge theory confines. Although in this case confinement is due to kinematics, this finding is likely to be the weak coupling equivalent of confinement in gauge theories. For small but non-zero temperatures, the Polyakov loop is still zero if the theory has an unbroken center symmetry, but acquire a non-zero (albeit exponentially small $\mathcal{O}(e^{-\beta/R})$) vacuum expectation value otherwise. The very same features characterise the high temperature phase, although their explicit realisation differs from the low temperature

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[4]In the $A_0 = 0$ gauge. In a normalization of fields that correspond to canonical kinetic terms in the action, this corresponds to taking the $g \to 0$ limit after rescaling $A_i \to \frac{1}{g}A_i$.

regime. In the case in which the system is center symmetric, this symmetry cannot be broken on a finite volume, hence the Polyakov loop will still be zero at high temperature. However, this zero is due to a different phenomenon, namely the emergence of center-breaking minima of the effective potential, over which we have to take an average when we are not the in the thermodynamic limit. In this limit one minimum is selected, and this gives a non-zero value to the Polyakov loop. This is the case for $SU(N)$ and $SO(2N)$ gauge theories. For centerless theories, like G_2 and $SO(2N+1)$, there seems to be only one minimum for the effective potential in the high temperature phase. On this saddle point, the Polyakov loop is different from zero.

In this paper, we have not considered Polyakov loop in the spinor representations of $SO(N)$. Since these Polyakov loops will be non-trivially charged under the center of the gauge group, they should exhibit a behavior similar to the fundamental Polyakov loop in the $SU(N)$ theory, i.e. at finite N, they should be zero at all temperatures. In the thermodynamic (large N) limit, after restriction to one center-breaking minimum, we should find a non-zero value of the Polyakov loop. We have also not studied the Polyakov loop in the adjoint representation of $SU(N)$ theories. We expect to find that all representations uncharged under the center would have a non-zero Polyakov loop at non-zero temperature with a smooth crossover between the low temperature and the high temperature regimes.

The minima of the effective potential exhibit a different structure in the low and the high temperature phase. This is true independently of the existence of an unbroken center group of the theory. This means that with a finite number of degrees of freedom, a cross over between the confined phase and the deconfined phase should take place. In the large N limit, this cross-over becomes a real phase transition. For finite N or in the case of exceptional groups, whether there is a cross-over or a real phase transition cannot be established by a computation on $S^3 \times S^1$. Recent lattice calculations suggest that in pure gauge theories the change of properties of the vacuum is always associated with a proper phase transition [17, 21 – 23].

Another interesting question is whether calculations on a $S^3 \times S^1$ manifold can shed some light over the mechanism of color confinement. For QCD with matter in the adjoint representation, an interesting development in this direction is reported in [24].

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